Triangle Center

A triangle center (sometimes simply called a center) is a point whose trilinear coordinates are defined in terms of the side lengths and angles of a triangle and for which a triangle center function can be defined. The function giving the coordinates $\alpha : \beta : \gamma$ is called the triangle center function. The four ancient centers are the triangle centroid, incenter, circumcenter, and orthocenter.

The triangle center functions of triangles centers therefore satisfy homogeneity

$$f(t\,a, t\,b, t\,c) = t^n f(a, b, c), \quad (1)$$

bisymmetry in b and c,

$$f(a, c, b) = f(a, b, c) \quad (2)$$

and cyclicity in a, b, and c,

$$\alpha : \beta : \gamma = f(a, b, c) : f(b, c, a) : f(c, a, b) \quad (3)$$

(Kimberling 1998, p. 46).

Note that most, but not all, special triangle points therefore qualify as triangle centers. For example, bicentric points fail to satisfy bisymmetry, and are therefore excluded. The most common examples of points of this type are the first and second Brocard points, for which triangle center-like functions can be defined that obey homogeneity and cyclicity, but

not bisymmetry.

Note also that it is common to give triangle center functions in an abbreviated form $f'(a, b, c)$ that does not explicitly satisfy bisymmetry, but rather biantisymmetry, so $f'(a, c, b) = -f'(a, b, c)$. In such cases, $f'(a, b, c)$ can be converted to an equivalent form $f(a, b, c)$ that *does* satisfy the bisymmetry property by defining

$$f(a, b, c) = [f'(a, b, c)]^2 f'(b, c, a) f'(c, a, b). \tag{4}$$

An example of this kind is Kimberling center X_{100}, which has a tabulated center of

$$\alpha_{100} = \frac{1}{b - c}, \tag{5}$$

which corresponds to the true triangle center function

$$\alpha_{100} = \frac{1}{(a - b)(b - c)^2 (c - a)}. \tag{6}$$

A triangle center is said to be polynomial iff there is a triangle center function f that is a polynomial in a, b, and c (Kimberling 1998, p. 46).

Similarly, a triangle center is said to be regular iff there is a triangle center function f that is a polynomial in Δ, a, b, and c, where Δ is the area of the triangle).

A triangle center is said to be a major triangle center if the triangle center function $\alpha = f(A, B, C)$ is a function of angle A alone, and therefore β and γ of B and C alone, respectively.

C. Kimberling (1998) has extensively tabulated triangle centers and their trilinear coordinates, assigning a unique integer to each. In this work, these centers are called Kimberling centers, and the nth center is denoted X_n, the first few of which are summarized below.

X_n	center	triangle center function α

X_1	incenter I	1
X_2	triangle centroid G	$1/a$, bc, $\csc A$
X_3	circumcenter O	$\cos A$, $a(b^2+c^2-a^2)$
X_4	orthocenter H	$\sec A$
X_5	nine-point center Ni	$\cos(B-C)$, $\cos A + 2\cos B\cos C$, $bc\left[a^2b^2+a^2c^2-(b^2-c^2)^2\right]$
X_6	symmedian point K	a, $\sin A$
X_7	Gergonne point Ge	$bc/(b+c-a)$, $\sec^2\left(\frac{1}{2}A\right)$
X_8	Nagel point Na	$(b+c-a)/a$, $\csc^2\left(\frac{1}{2}A\right)$
X_9	mittenpunkt M	$b+c-a$, $\cot\left(\frac{1}{2}A\right)$
X_{10}	Spieker center Sp	$bc(b+c)$
X_{11}	Feuerbach point F	$1-\cos(B-C)$, $\sin^2((B-C)/2)$
X_{12}	harmonic conjugate of X_{11} with respect to X_1 and X_5	$1+\cos(B-C)$, $\cos^2((B-C)/2)$, $bc(b+c)^2/(b+c-a)$
X_{13}	first Fermat point X	$\csc(A+\pi/3)$, $\sec(A-\pi/6)$
X_{14}	second Fermat point X'	$\csc(A-\pi/3)$, $\sec(A+\pi/6)$
X_{15}	first isodynamic point S	$\sin(A+\pi/3)$, $\cos(A-\pi/6)$
X_{16}	second isodynamic point S'	$\sin(A-\pi/3)$, $\cos(A+\pi/6)$
X_{17}	first Napoleon point N	$\csc(A+\pi/6)$, $\sec(A-\pi/3)$
X_{18}	second Napoleon point N'	$\csc(A-\pi/6)$, $\sec(A+\pi/3)$
X_{19}	Clawson point	$\tan A$, $a\sec A$, $1/(b^2+c^2-a^2)$, $\sin(2B)+\sin(2C)-\sin(2A)$
X_{20}	de Longchamps point L	$\cos A - \cos B\cos C$

E. Brisse has compiled a separate list of 2001 triangle centers.

SEE ALSO: Areal Coordinates, Barycentric Coordinates, Exact Trilinear Coordinates, Kimberling Center, Major Triangle Center, Polynomial Triangle Center, Regular Triangle Center, Triangle, Triangle Center Function, Trilinear Coordinates, Trilinear Polar. [Pages Linking Here]

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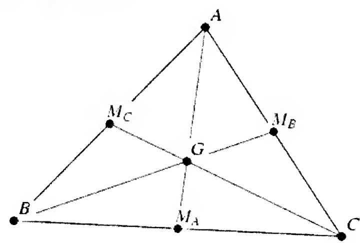
LAST MODIFIED: October 25, 2003

CITE THIS AS:

Weisstein, Eric W. "Triangle Center." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/TriangleCenter.html

Triangle Centroid



The geometric centroid (center of mass) of the polygon vertices of a triangle is the point G (sometimes also denoted M) which is also the intersection of the triangle's three triangle medians (Johnson 1929, p. 249; Wells 1991, p. 150). The point is therefore sometimes called the median point. The centroid is always in the interior of the triangle. It has equivalent triangle center functions

$$\alpha = \frac{1}{a} \quad (1)$$

$$\alpha = b\,c \quad (2)$$

$$\alpha = \csc A, \quad (3)$$

and homogeneous barycentric coordinates $(1, 1, 1)$. It is Kimberling center X_2.

The centroid satisfies

$$A G^2 + B G^2 + C G^2 = \frac{1}{3}(a^2 + b^2 + c^2). \tag{4}$$

The centroid of the triangle with trilinear vertices $p_i : q_i : r_i$ for $i = 1, 2, 3$ is given by

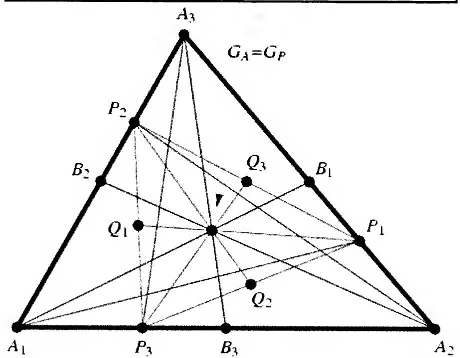
$$\begin{array}{c} \frac{p_1}{a p_1 + b q_1 + c r_1} + \frac{p_2}{a p_2 + b q_2 + c r_2} + \frac{p_3}{a p_3 + b q_3 + c r_3} : \\ \frac{q_1}{a p_1 + b q_1 + c r_1} + \frac{q_2}{q_2 + q_2 + r_2} + \frac{q_3}{a p_3 + b q_3 + c r_3} : \\ \frac{r_1}{a p_1 + b q_1 + c r_1} + \frac{r_2}{a p_2 + b q_2 + c r_2} + \frac{r_3}{a p_3 + b q_3 + c r_3} \end{array} \tag{5}$$

(P. Moses, pers. comm., Sep. 7, 2005).

The following table summarizes the triangle centroids for named triangles that are Kimberling centers.

triangle	Kimberling	triangle centroid
anticomplementary triangle	X_2	triangle centroid
circumnormal triangle	X_3	circumcenter
circumtangential triangle	X_3	circumcenter
contact triangle	X_{354}	Weill point
Euler triangle	X_{381}	midpoint of X_2 and X_4
excentral triangle	X_{165}	centroid of the excentral triangle
extouch triangle	X_{210}	X_{10} -Ceva conjugate of X_{37}
first Brocard triangle	X_2	triangle centroid
first Morley triangle	X_{356}	first Morley center
first Neuberg triangle	X_2	triangle centroid
incentral triangle	X_{1962}	bicentric sum of pu(32)

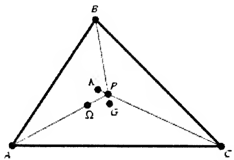
inner Napoleon triangle	X_2	triangle centroid
inner Vecten triangle	X_2	triangle centroid
medial triangle	X_2	triangle centroid
orthic triangle	X_{51}	centroid of orthic triangle
outer Napoleon triangle	X_2	triangle centroid
outer Vecten triangle	X_2	triangle centroid
reference triangle	X_2	triangle centroid
second Neuberg triangle	X_2	triangle centroid
Stammler triangle	X_3	circumcenter
tangential triangle	X_{154}	X_3 -Ceva conjugate of X_6

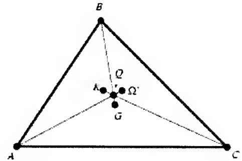


If the sides of a triangle $\Delta A_1 A_2 A_3$ are divided by points P_1, P_2, and P_3 so that

$$\frac{\overline{A_2 P_1}}{\overline{P_1 A_3}} = \frac{\overline{A_3 P_2}}{\overline{P_2 A_1}} = \frac{\overline{A_1 P_3}}{\overline{P_3 A_2}} = \frac{p}{q}, \qquad (6)$$

then the centroid G_P of the triangle $\Delta P_1 P_2 P_3$ is simply G_A, the centroid of the original triangle $\Delta A_1 A_2 A_3$ (Johnson 1929, p. 250).





One Brocard line, triangle median, and symmedian (out of the three of each) are concurrent, with $A\,\Omega$, $C\,K$, and $B\,G$ meeting at a point, where Ω is the first brocard point and K is the symmedian point. Similarly, $A\,\Omega'$, $B\,G$, and $C\,K$, where Ω' is the second Brocard point, meet at a point which is the isogonal conjugate of the first (Johnson 1929, pp. 268-269).

Pick an interior point X. The triangles $B\,X\,C$, $C\,X\,A$, and $A\,X\,B$ have equal areas iff X corresponds to the centroid. The centroid is located 2/3 of the way from each polygon vertex to the midpoint of the opposite side. Each median divides the triangle into two equal areas; all the medians together divide it into six equal parts, and the lines from the median point to the polygon vertices divide the whole into three equivalent triangles. In general, for any line in the plane of a triangle $A\,B\,C$,

$$d = \frac{1}{3}(d_A + d_B + d_C), \qquad (7)$$

where d, d_A, d_B, and d_C are the distances from the centroid and polygon vertices to the line.

A triangle will balance at the centroid, and along any line passing through the centroid. The trilinear polar of the centroid is called the Lemoine axis. The perpendiculars from the centroid are proportional to s_i^{-1},

$$a_1 p_2 = a_2 p_2 = a_3 p_3 = \frac{2}{3} \Delta, \tag{8}$$

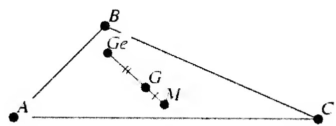
where Δ is the area of the triangle. Let P be an arbitrary point, the polygon vertices be A_1, A_2, and A_3, and the centroid G. Then

$$P A_1^2 + P A_2^2 + P A_3^2 = G A_1^2 + G A_2^2 + G A_3^2 + 3 P G^2. \tag{9}$$

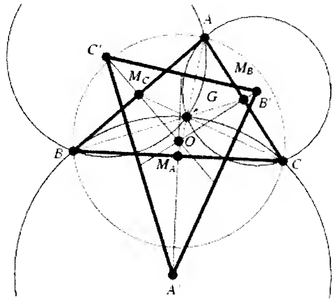
If O is the circumcenter of the triangle's centroid, then

$$O G^2 = R^2 - \frac{1}{9} (a^2 + b^2 + c^2). \tag{10}$$

The centroid lies on the Euler line and Nagel line. The centroid of the perimeter of a triangle is the triangle's Spieker center (Johnson 1929, p. 249). The symmedian point of a triangle is the centroid of its pedal triangle (Honsberger 1995, pp. 72-74).



The Gergonne point Ge, triangle centroid G, and mittenpunkt M are collinear, with $Ge\,G : G\,M = 2:1$.



Given a triangle ΔABC, construct circles through each pair of vertices which also pass through the triangle centroid G. The triangle $\Delta A'B'C'$ determined by the center of these circles then satisfies a number of interesting properties. The first is that the circumcircle O and triangle centroid G of ΔABC are, respectively, the triangle centroid G' and symmedian point K' of the triangle $\Delta A'B'C'$ (Honsberger 1995, p. 77). In addition, the triangle medians of ΔABC and $\Delta A'B'C$ intersect in the midpoints of the sides of ΔABC.

SEE ALSO: Circumcenter, Euler Line, Exmedian Point, Incenter, Nagel Line, Orthocenter. [Pages Linking Here]

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LAST MODIFIED: November 3, 2005

CITE THIS AS:

Weisstein, Eric W. "Triangle Centroid." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/TriangleCentroid.html